**Week 5 (Day 1)**

(4 Oct. 2016)

Covered:

- Four rules of derivatives (i.e. $+, -, \times, \div$)
- Mentioned Chain Rule (i.e. derivative of composite function of two functions)

**Four rules of derivatives**

Assumption: In the following let $f(x), g(x)$ be two functions, both having the same domain, and both differentiable at the point $x = c$ in the domain. Then we have (*) $f(x) \pm g(x)$, $f(x)g(x)$, $f(x)/g(x)$ are all differentiable at $x = c$. (For the last one, one has to make the extra assumption that $g(c) \neq 0$.)

Furthermore, the derivatives of these "sum", "difference", "product" and "quotient" functions at the point $x = c$ are given by formulas listed below:

1. The derivative of the sum function $f(x) + g(x)$ at $x = c$ (If you like, you can give a name to this function, calling it for example $(f + g)(x)$ or $h(x)$ ) has the following formula.

$$\left.\frac{d(f(x) + g(x))}{dx}\right|_{x=c} = f'(c) + g'(c)$$

2. Similarly, for the function $f(x) - g(x)$, we have

$$\left.\frac{d\big(f(x) - g(x)\big)}{dx}\right|_{x=c} = f'(c) - g'(c)$$

3. (Product Rule) For product of these two functions, the formula is slightly different, i.e.

$$\left.\frac{d(f(x)g(x))}{dx}\right|_{x=c} = g(c)f'(c) + g'(c)f(c)$$

**Remark:** In the case when $g(x) \equiv k$ (i.e. it is constantly equal to $k$), the above formula has the simpler form

$$\left.\frac{d(kf(x))}{dx}\right|_{x=c} = kf'(c)$$

4. (Quotient Rule) For quotient, it is

$$\left.\frac{d\big(f(x)g(x)\big)}{dx}\right|_{x=c} = \frac{f'(c)g(c) - g'(c)f(c)}{\big(g(c)\big)^2}$$

**Idea of Proof of the Product Rule**

We just outline one or two of the ideas. If you are interested in more detail, just send me an e-mail. I will explain more to you.

**A Preparatory Theorem**

To show the product rule, we need the following "little" result:

**Theorem** (Differentiable at $x = c$ $\Rightarrow$ continuous at $x = c$.)

Assume $f(x)$ is differentiable at $x = c$, then $f(x)$ is continuous at $x = c$.

**Proof:**

Main idea is to start from the statement $\lim_{h \to 0} f(c + h) = f(c)$ (definition of "continuous at $x = c$.") and try to connect it to i.e. $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$ (definition of "differentiable at $x = c$.").

The connection can be established if one looks at the expressions:

(i) $f(c + h) - f(c)$ and

(ii) $\frac{f(c+h)-f(c)}{h}$

This is because $f(c + h) - f(c) = \frac{(f(c+h)-f(c))}{h} \cdot h$

Now we know that in the above equation, both of the limits $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ and the limit $\lim_{h \to 0} h$ exist.

Furthermore, the first of them is equal to $f'(c)$, which is a finite number. The second one is equal to zero.

Combining all these, we get for the right-hand side:

$$\lim_{h \to 0} \frac{(f(c + h) - f(c))}{h} \cdot \lim_{h \to 0} h = f'(c) \cdot 0 = 0$$

It follows that the limit of the left-hand side also exists and is given by

$$\lim_{h\to 0}\big(f(c+h)-f(c)\big)=\lim_{h\to 0}\left(\frac{\big(f(c+h)-f(c)\big)}{h}h\right)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\cdot\lim_{h\to 0}h$$
$$=0$$

**Steps of the Proof of Product Rule**

1. Consider the "Difference Quotient" i.e.

$$\frac{f(c+h)g(c+h)-f(c)g(c)}{h}$$

2. Rewrite it in the form (because we only know the following limits to exist: (i) $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$, (ii) $\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}$ ):

$$\frac{f(c+h)g(c+h)-f(c+h)g(c)+f(c+h)g(c)-f(c)g(c)}{h}$$

Grouping terms we get from the above:

$$\frac{f(c+h)[g(c+h)-g(c)]}{h}+g(c)\,\frac{f(c+h)-f(c)}{h}$$

3. Take limit $h\to 0$. The term $\frac{g(c+h)-g(c)}{h}$ goes to the limit $g'(c)$. On the other hand, the term $\frac{f(c+h)-f(c)}{h}$ goes to the limit $f'(c)$. (You can write these two facts in the form: $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)$ and $\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}=g'(c)$ ).

4. We still have two more limits to consider. They are:

(i) $\lim_{h\to 0}f(c+h)$ and

(ii) (ii) $\lim_{h\to 0}g(c)$.

Since $f(x)$ is <u>differentiable</u> at $x=c$, it is <u>continuous</u> at $x=c$. So the first one is just $\lim_{h\to 0}f(c+h)=f(c)$ . As for the second one, $g(c)$ is a constant function, so its limit is given by $\lim_{h\to 0}g(c)=g(c)$.

5. Combining all the above, we get

$$\lim_{h\to 0}f(c+h)\,\lim_{h\to 0}\frac{[g(c+h)-g(c)]}{h}+g(c)\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$
$$=f(c)g'(c)+g(c)f'(c).$$

Friday Group math1510

**Week 5 (Day 2)**

(7 Oct. 2016)

Done:

- Chain Rule, application, proof strategy
- Implicit differentiation
- Intermediate Value Theorem

**An Application of the Chain Rule**

Show that $\frac{dx^\alpha}{dx} = \alpha x^{\alpha-1}$, for any $\alpha \in R$ and any $x > 0$.

(Solution)

Interpret $x^\alpha$ as $e^{\alpha \ln x}$ (where $x > 0$).

Then using Chain Rule, we obtain (by letting $y = \alpha \ln x$) [ For simplicity, we don't write $|_{x=c}$ here.]

$$\frac{dx^\alpha}{dx} = \frac{d(e^{\alpha \ln x})}{dx} = \frac{d(e^{\alpha \ln x})}{dy} \frac{d(\alpha \ln x)}{dx}$$

$$= \frac{d(e^y)}{dy} \frac{d(\alpha \ln x)}{dx} = e^y \cdot \alpha \left(\frac{1}{x}\right) = e^{\alpha \ln x} \cdot \alpha \cdot x^{-1}$$

$$= e^{\ln x^\alpha} \cdot \alpha \cdot x^{-1} = x^\alpha \alpha x^{-1} = \alpha x^{\alpha-1}.$$

**Mathematical Formulation of Chain Rule**

The chain rule say:

Theorem. If $f(y)$ and $g(x)$ are two functions. Assume

(i) $f(y)$ is differentiable at $y = g(c)$ ;

(ii) $g(x)$ is differentiable at $x = c$.

Then

(i) $f(g(x))$ is differentiable at $x = c$ and

(ii) $\left.\frac{df(g(x))}{dx}\right|_{x=c} = f'(g(c)) \cdot g'(c)$.

Proof Strategy

Again 2 steps.

(Step 1) Consider the Difference Quotient, i.e.

$$\frac{f(g(c+h)) - f(g(c))}{h}$$

Rewrite it as

$$\frac{f\big(g(c+h)\big)-f\big(g(c)\big)}{g(c+h)-g(c)}\cdot\frac{g(c+h)-g(c)}{h}$$

Having done this, take limit and let $h \to 0$ in the above expression. This will lead to $f'\big(g(c)\big)g'(c)$

**Remark:** Two things to note.

1. The prime (i.e. $'$ ) in $f'\big(g(c)\big)$ means "differentiation with respect to $y$", while the prime (i.e $'$) in $g'(x)$ means "differentiation with respect to $x$".
2. The above "proof strategy" has a lot of things which one has to fix. For example, one has to consider what happens if $g(c+h)-g(c)=0$ for numbers $c+h$ near to $c$.

**Implicit Differentiation**

In high schools, you may have learned this way of computing derivative of a function:

$$x^2+y^2=a^2$$

Then compute the derivative of $x$, then of $y$, then of $a$ (which is on the right-hand side of the equation) all with respect to the independent variable $x$. Having done this, we obtain

$$\frac{dx^2}{dx}+\frac{dy^2}{dx}=\frac{da^2}{dx}$$

Now $\frac{dx^2}{dx}=2x$, $\frac{dy^2}{dx}=\frac{dy^2}{dy}\frac{dy}{dx}=2y\cdot y'$ and $\frac{da^2}{dx}=0$

Result: We get now $2x+2yy'=0$ implying $y'=\frac{-x}{y}$.

Remark: To compute the value of this derivative, we need two numbers, i.e. both $x$ and $y$. Or we can express $y$ in terms of $x$ using the equation

$$x^2+y^2=a^2$$

to get $y'=-\frac{x}{\pm\sqrt{a^2-x^2}}=\mp\left(\frac{x}{\sqrt{a^2-x^2}}\right)$.

Question: Why can we do this?

Answer: This is due to the

Implicit Function Theorem, which roughly says:
Given any function of two variables $x$ and $y$, i.e. $f(x, y)$ and an equation $f(x, y) = c$ (the right-hand side is a constant), then we have

1. $y$ is a function of $x$ or
2. $x$ is a function of $y$.

In symbols, we write the sentence "$y$ is a function of $x$" as "$y = y(x)$". (We don't write things like "$y = f(x)$" because that would need an extra letter $f$.)

Similarly, the second sentence becomes $x = x(y)$.

**Example:**
Suppose $f(x, y) = \cos(xy)\, e^{xy}$.
Now the above theorem says $f(x, y) = c$ implies $y = y(x)$ or $x = x(y)$. Let's suppose the first case is true, then we can find $\frac{dy}{dx}$.

Doing this, we obtain

$$\frac{d \cos(xy)\, e^{xy}}{dx} = \frac{dc}{dx} = 0$$

The left-hand side is: $\frac{d\cos(xy)}{dx} e^{xy} + \cos(xy) \frac{de^{xy}}{dx} = 0$

$$-\sin(xy)\frac{d(xy)}{dx} e^{xy} + \cos(xy)\left[e^{xy}\left(\frac{d(xy)}{dx}\right)\right] = 0$$

$$-\sin(xy)\left\{x\frac{dy}{dx} + \frac{dx}{dx}y\right\} e^{xy} + \cos(xy)\left[e^{xy}\left\{x\frac{dy}{dx} + \frac{dx}{dx}y\right\}\right] = 0$$

$$-\sin(xy)\{xy' + y\}e^{xy} + \cos(xy)\left[e^{xy}\{xy' + y\}\right] = 0$$

$$y' x e^{xy}\left[-\sin(xy) + \cos(xy)\right] = y e^{xy}\left[\sin(xy) - \cos(xy)\right]$$

Hence $y' = -\frac{y}{x}$ after writing $y'$ on the left-hand side and the rest on the right-hand side.

**Remark:** We said "roughly" because the theorem requires some "differentiability" conditions on the function $f(x, y)$ which is usually satisfied. Also, the “or” can mean “either/or” or “both”.

**(Optional)**

In more advanced books, you can find the following statement:

$$f(x, y) = c \quad \text{gives after differentiating with respect to } x,$$

$$\frac{df(x,y)}{dx} = \frac{df(x, y(x))}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial x}\frac{dy(x)}{dx}$$

Similarly, when differentiated with respect to $y$, it gives

$$\frac{df(x,y)}{dy} = \frac{df(x(y), y)}{dy} = \frac{\partial f}{\partial x}\frac{dx(y)}{dy} + \frac{\partial f}{\partial y}\frac{dy}{dy}$$

The expressions $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are called “partial derivatives”. We will talk about them later in the lectures.

**Intermediate Value Theorem**

This is a useful consequence of the continuity of a function in an interval. More precisely, we have

**Theorem.**

Suppose $f: [a, b] \to R$ is a function whose domain is the closed interval (i.e. the end-points $a$ and $b$ are included).

Suppose also that $f(x)$ is continuous at each point in $[a, b]$, and that $f(a) \cdot f(b) < 0$. (in other words, $f(a)$ and $f(b)$ are of “different” signs!) Then the curve $y = f(x)$ must intersect the $x$ −axis somewhere between $a$ and $b$.

(In other words, the equation $f(x) = 0$ has a solution (maybe more than 1!) in the interval $(a, b)$.)

**Remark:** The end-points cannot be solution of this equation, because we are assuming that $f(a) < 0$ or $> 0$ (and correspondingly $f(b) > 0$ or $< 0$).

**Application:**

The polynomial equation: $x^7 + 100x^4 + 13x + 17 = 0$ has a solution.

Idea of solution: Let $f(x) = x^7 + 100x^4 + 13x + 17$
Find $a$ and $b$ so that the $f(a) < 0$ and $f(b) > 0$. Then by the above theorem, the equation $f(x) = 0$ has a solution in $[a, b]$.